Control of entanglement in a closed, 3-qubit system interacting via an isotropic Heisenberg Hamiltonian

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The time-evolution of the entanglement is studied for a closed system consisting of 3 qubits with two of the pairs of qubits interacting via an isotropic Heisenberg Hamiltonian. Even the non-interacting pair of qubits becomes entangled and can be projected onto one of the Bell states with probability one.

1. Introduction

The possibility of entanglement of physical systems is perhaps the most unexpected consequence of quantum mechanics. The advent of quantum information theory shows that entanglement is also useful [1]. Entanglement can be generated by the Heisenberg interaction between spins and this is of practical importance because the Heisenberg interaction applies to real physical systems, e.g. the controllable interaction between quantum dots, which have great potential for implementing quantum information and quantum computing schemes [1,2].

In this work we study the Heisenberg interaction between three qubits which offer advantages over two-qubit Bell states for quantum information purposes [3]. Most previous studies involving few- or many-qubits have considered thermal entanglement and imposed periodic boundary conditions. Since the aim in quantum information applications is to avoid decoherence, we consider entanglement in the ground state, rather than thermal entanglement, and avoid imposing periodic boundary conditions which are unnecessarily restrictive and physically unreasonable for few-qubit applications.

2. Isotropic Heisenberg model with inhomogeneous magnetic field

It is unnecessary to consider the anisotropic Heisenberg Hamiltonian because the anisotropy is small in practice or can be avoided [1,2]. On the other hand, inclusion of an inhomogeneous magnetic field accounts for the inhomogeneities in real sample systems as well as the likely presence of stray magnetic fields due to magnetic impurities in the sample [2]. As shown in Fig. 1, we study the case where there is no direct interaction between the spins labelled 1 and 2 and the magnetic field is non-zero on only the site labelled 3.

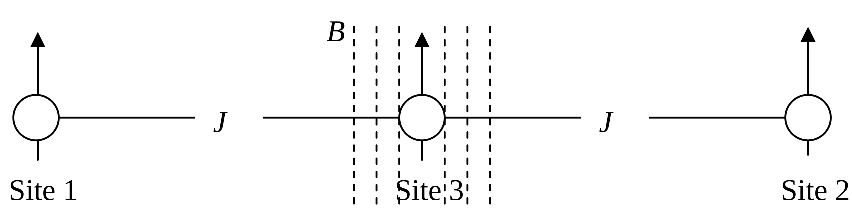

Fig. 1. Three qubits showing interaction between sites 1 and 3, and 3 and 2, and possible magnetic field at site 3.

The interaction between the 3 qubits can be described by the Hamiltonian

$$H = \frac{J}{2}(\sigma_1 \cdot \sigma_3 + \sigma_3 \cdot \sigma_2) + B\sigma_3^z \tag{1}$$

where J and B are constants with the units of energy and the spin on site l is $\mathbf{S}_l = \boldsymbol{\sigma}_l / 2$ in units of $\hbar$. The Hamiltonian commutes with $S_1^z + S_2^z + S_3^z$ and we will be interested only in those states with total z-component of 1/2 (in units of $\hbar$). In terms of the usual notation where, for example, $|101\rangle$ represents spins on sites 1 and 3 up in the z –direction, and the spin on site 2 down in the z –direction, the relevant eigenvalues and eigenstates of the above Hamiltonian are

$$\begin{aligned} E_1 &= B & |\psi_1\rangle &= \frac{1}{\sqrt{2}}\left(|101\rangle - |011\rangle\right) \\ E_2 &= -\frac{J}{2}(y+1) & |\psi_2\rangle &= \frac{1}{\sqrt{2y\alpha}}\left[\alpha|110\rangle - 2\left(|101\rangle + |011\rangle\right)\right] \\ E_3 &= \frac{J}{2}(y-1) & |\psi_3\rangle &= \frac{1}{\sqrt{2y\beta}}\left[\beta|110\rangle + 2\left(|101\rangle + |011\rangle\right)\right] \end{aligned} \quad (2)$$

where $y = \sqrt{4x^2 + 4x + 9}$, $x = B/J$, $\alpha = y + 2x + 1$, $\beta = y - 2x - 1$.

If the three qubits are prepared in the state $|\varphi(0)\rangle = |110\rangle$ at time t = 0, then at time t,

$$\begin{aligned} |\varphi(t)\rangle &= \frac{1}{\sqrt{2y}}\left(\sqrt{\alpha}e^{-iE_2 t/\mathrm{h}}|\psi_2\rangle + \sqrt{\beta}e^{-iE_3 t/\mathrm{h}}|\psi_3\rangle\right) \\ &= \frac{1}{2y}\left[\left(\alpha e^{-iE_2 t/\mathrm{h}} + \beta e^{-iE_3 t/\mathrm{h}}\right)|110\rangle - 2\left(e^{-iE_2 t/\mathrm{h}} - e^{-iE_3 t/\mathrm{h}}\right)\left(|101\rangle + |011\rangle\right)\right]. \end{aligned} \quad (3)$$

3. Entanglement

The entanglement of any pair of the three qubits can be quantified by the concurrence $C(\mathrm{ij}) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$ where $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ are the (positive) square roots of the eigenvalues, arranged in decreasing order, of the operator $\varsigma(ij) = \rho_{ij}\left(\sigma_i^y \otimes \sigma_j^y\right)\rho^*_{ij}\left(\sigma_i^y \otimes \sigma_j^y\right)$ where $\rho_{ij} = \mathrm{Tr}_k \rho_{ijk}$ is the partial trace of the density operator of the three qubits [4]. For pure states (as considered here), the entanglement of all three qubits can be quantified by the three-tangle $\tau(ijk) = 2\left(\lambda_1^{ij}\lambda_2^{ij} + \lambda_1^{ik}\lambda_2^{ik}\right)$ where $\lambda_1^{ij}, \lambda_2^{ij}$ are the square roots of the two non-zero eigenvalues of $\varsigma(ij)$ (the other two eigenvalues being zero for pure states) [4].

The three-tangle remains zero at all times but each of the three pairs of qubits become entangled as shown by the following expressions for the concurrences

$$\begin{aligned} C(12) &= 2y^{-2}\left(1 - \cos\omega t\right) \\ C(23) = C(12) &= \sqrt{2}y^{-2}\sqrt{\left(1 - \cos\omega t\right)\left(y^2 - 4(1 - \cos\omega t\right)} \\ \text{where} \quad \omega &= \left(E_3 - E_2\right)/\mathrm{h} = yJ/\mathrm{h}. \end{aligned} \quad (4)$$

The concurrences when the magnetic field is zero are shown as a function of time in Fig. 2.

3. Production of a Bell state

It is interesting that the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle_{12} + |01\rangle_{12}\right) \quad (5)$$

involving the spins on sites 1 and 2, which have no direct interaction with each other, can be produced by simply measuring the z-component of the spin on site 3 repeatedly at arbitrary time intervals until the result spin up on site 3 is obtained. This follows from Eq. (3) because the result spin up for the spin on site 3 is obtained with probability $2y^{-2}(1-\cos\omega t)$ in which case the spins on sites 1 and 2 are projected onto the Bell state $\left|\Phi^{+}\right\rangle_{12}$ in Eq. (5) and they remain in that state if the controllable interaction between the qubits is turned off when the result is obtained. On the other hand if the result spin down on site 3 is obtained, the three qubits are projected onto the state $|110\rangle$ which was the original state at time $t = 0$ and so the process can be repeated until the spin at site 3 is found to be up, in which case $\left|\Phi^{+}\right\rangle_{13}$ is obtained as explained above.

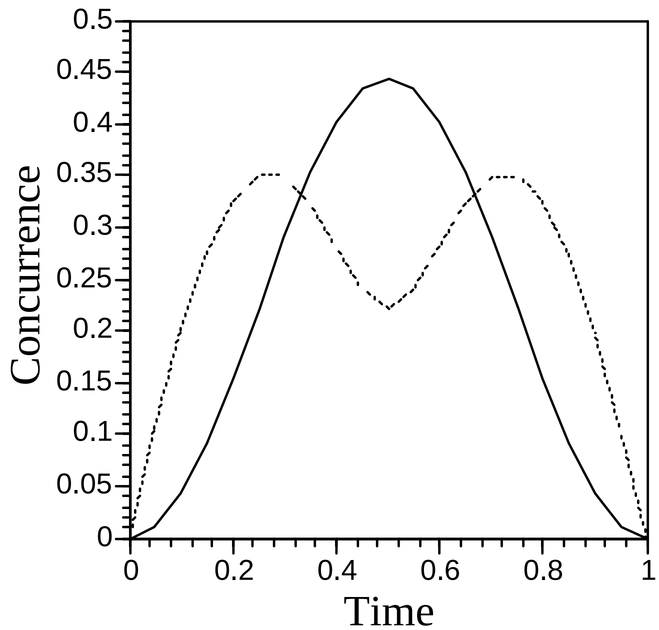

Fig. 2. The concurrence as a function of time *t* in units of *h/yJ* in the absence of the magnetic field. The full line shows the concurrence *C*(12) of the spins on sites 1 and 2. The dotted line shows *C*(13) = *C*(32).

5. Discussion and Conclusion

When two qubits (sites 1 and 2) are allowed to interact with a third qubit (site 3) via the isotropic Heisenberg interaction (Eq. (1), which includes the possibility of a magnetic field on site 3), the three qubits remain unentangled as a whole (three-tangle remains zero) but the qubits become entangled in pairs. As shown in Fig. 2 and Eq. (4), the entanglement as measured by each pairwise concurrence fluctuates in time with frequency which depends on the strength of the Heisenberg interaction *J* and also depends the applied magnetic field through $x = B/J$ in the term *y*. The magnitudes of the concurrences also depend on $x = B/J$ in the term *y*. The maximum concurrence for qubits 1 and 2 occurs when $B = 0$ and it is 4/9. The interaction which has been considered also provides a means of producing a Bell state involving the non-interacting qubits 1 and 2.

References

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