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Alternative Experimental Protocol for a PBR-Like Theorem

D. J. Miller

Centre for Time, University of Sydney, Sydney 2006 and
School of Physics, University of New South Wales, Sydney 2052.

Pusey, Barrett and Rudolph (PBR) have shown that "distinct quantum states must correspond to physically distinct states of reality." The PBR protocol is adapted here to apply to three states rather than two. Only two sources are required and spin-spin interactions are sufficient for a weaker PBR-like result.

1. Introduction

Recently Pusey, Barrett and Rudolph (PBR) [1] proved a significant new constraint on the interpretation of quantum states. Their result "may be the most important general theorem relating to the foundations of quantum mechanics since Bell's theorem" [2] or at least in "the past couple of years" [3]. On one view, e.g. expressed by Bohr and Heisenberg at times, a quantum state $|u\rangle$ provides a means of calculating the probabilities of macroscopic events and cannot be related to properties of physical systems at the quantum level. PBR is not relevant to that interpretation of $|u\rangle$. Another view (the "statistical" interpretation of quantum mechanics), e.g. expressed by Einstein, likens $|u\rangle$ to a probability distribution $u(x,p)$ of states in phase space in classical physics. The idea is that experimental outcomes are determined by a physical state, traditionally represented by λ. We do not have access to λ but only to $|u\rangle$ which is associated with a distribution $u(\lambda)$ over the λ and this is why $|u\rangle$ determines only the probabilities of experimental outcomes.

PBR show that this statistical interpretation of $|u\rangle$ is untenable. The PBR result involves two quantum states $|u\rangle$ and $|v\rangle$. If $|\langle u|v\rangle|^2 \leq ½$, $N=2$ independent sources of the states are sufficient but as $|\langle u|v\rangle|^2 \to 1$, $N \to \infty$. The PBR protocol also requires an N-bit entangling gate operating without post-selection. We show that a weaker PBR-like conclusion can be achieved without requiring an entangling gate and needing only $N=2$.

2. Basis of the PBR theorem

We will be concerned with up to three quantum states $|u\rangle$, $|v\rangle$ and $|w\rangle$ in Hilbert space. According to the statistical interpretation (assumed to be correct in this section), there are three associated distributions $u(\lambda)$, $v(\lambda)$ and $w(\lambda)$ over the space Λ of the physical states λ. As shown in Fig. 1(a) and (b), two distributions $u(\lambda)$ and $v(\lambda)$ may be (i) "disjoint" so that no state λ lies in the support of both $u(\lambda)$ and $v(\lambda)$ or (ii) "conjoint" in the sense that one or more of the physical states λ lies in the support of both $u(\lambda)$ and $v(\lambda)$, respectively.

For observable $\hat{U}$ with $\hat{U}|u\rangle = u|u\rangle$, a measurement of $\hat{U}$ on the state $|v\rangle$ will sometimes yield the outcome u if $\langle v|u\rangle \neq 0$. On the statistical interpretation, both $u(\lambda)$ and $v(\lambda)$ might contain some of the physical states from the set $\{\lambda\}_u$ which lead to the result u, i.e. $u(\lambda)$ and $v(\lambda)$ might be conjoint. PBR show the statistical interpretation is untenable by proving, in our terms, the *Two-State (PBR) Theorem*: The distributions $u(\lambda)$ and $v(\lambda)$ of any two states $|u\rangle$ and $|v\rangle$ in quantum mechanics are disjoint.

To avoid the experimental difficulties with the PBR protocol (see Sec. 1), we prove,

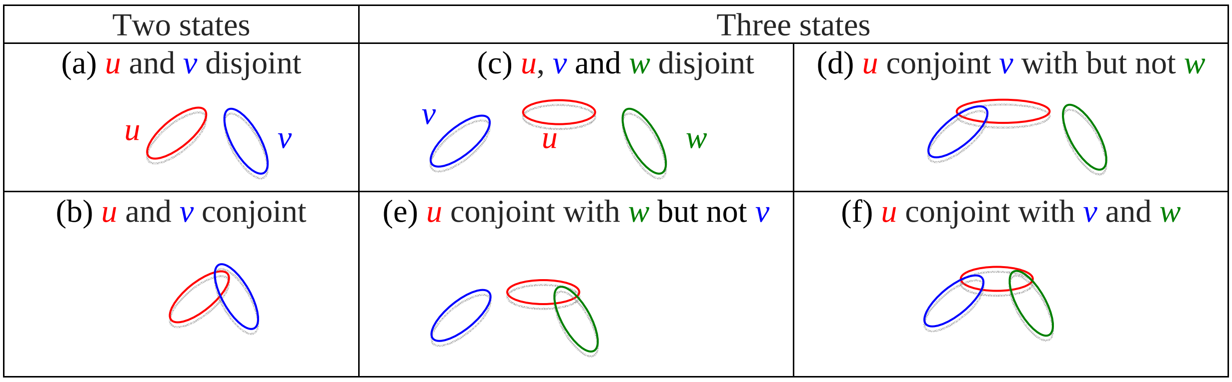

Fig. 1. The possible "joining" relationships for two states and the relevant cases for three states.

using an experimentally simpler protocol, a weaker result than PBR involving three states that we will call the *Three-State Theorem:* For any two states $|u\rangle$ and $|v\rangle$, there are two states $|w_n\rangle$, $n=\pm$, in the 2D subspace spanned by $|u\rangle$ and $|v\rangle$ with $\langle u|w_+\rangle \neq 0$ and $\langle v|w_-\rangle = 0$, and if $u(\lambda)$ is conjoint with $v(\lambda)$, $u(\lambda)$ is disjoint with both $w_+(\lambda)$ and $w_-(\lambda)$.

We now present the idea of the PBR proof modified to deal with our three-state theorem. Consider an experiment in which Alice prepares a system in the state $|u\rangle_A$ or $|v\rangle_A$ with corresponding distributions u_A (short for $u_A(\lambda)$) and v_A which are conjoint. Bob independently prepares a system in the state $|u\rangle_B$ or $|w\rangle_B$ with corresponding distributions that are also conjoint ($|u\rangle_B$ is the same state in Bob's space as $|u\rangle_A$ is in Alice's space; $|w\rangle \equiv |w_+\rangle$ and $|w\rangle \equiv |w_-\rangle$ require entirely different experimental set-ups, see Sec. 3). The joint distributions are $u_A u_B$, $u_A w_B$, $v_A u_B$ and $v_A w_B$. Fig. 1(c-f) show the "join" possibilities that are relevant to the present argument. Consider an experiment with four outcomes: R_1, R_2, R_3 and R_4 in which $u_A u_B$ never gives R_1, $u_A w_B$ never gives R_2, $v_A u_B$ never gives R_3 and $v_A w_B$ never gives R_4 - call this *Property PBR*. Consider a run of the experiment in which the physical state λ_A of Alice's system lies in the join region of u_A and v_A and λ_B of Bob's system lies in the join of $|u\rangle_B$ and $|w\rangle_B$. Given the preparations of Alice and Bob are independent, the joint state $\lambda = \lambda_A \lambda_B$ so it is possible that sometimes the joint state lies in the support of all four distributions $u_A u_B$, $u_A w_B$, $v_A u_B$ and $v_A w_B$. Because of *Property PBR*, if λ lies in the support of $w_A w_B$ it cannot lead to outcome R_1, if λ lies in the support of $w_A v_B$ it cannot lead to outcome R_2 and so on for R_3 and R_4. But this is impossible because one of those results obtains in any run of the experiment so, if there are experiments involving w_+ and w_- with *Property PBR*, we can say both (i) either *u* and *v* or u and w_+ (or both) are disjoint and (ii) either *u* and *v* or u and w_- (or both) are disjoint.

The remarkable result due to PBR [1] is an experiment which ensures *Property PBR* applies when (in our terms) $|v\rangle$ and $|w_+\rangle$ are the same state. So if there are statistical distributions corresponding to states, Fig. 1(a) applies to all. So there is a one-to-one correspondence between any quantum state and its own exclusive support in Λ and there is no point any more in saying any $|u\rangle$ corresponds to a statistical distribution $u(\lambda)$ over λ.

3. Proof of the three-state theorem

Since only a two-dimensional space is involved, one can picture $|u\rangle$, $|v\rangle$ and $|w_n\rangle$ as vectors on the Bloch sphere. For arbitrary $|u\rangle$ and $|v\rangle$, we can express $|v\rangle$ and construct $|w_n\rangle$ so that

$$|v\rangle = \cos\alpha|u\rangle + \sin\alpha|\bar{u}\rangle, \quad |w_n\rangle = n\sin\alpha|u\rangle + \cos\alpha|\bar{u}\rangle$$
$$\text{where } \langle\bar{u}|u\rangle = 0, \ \tan\alpha = \frac{|\langle\bar{u}|v\rangle|}{|\langle u|v\rangle|} = \frac{|\langle u|w_n\rangle|}{|\langle\bar{u}|w_n\rangle|}, \ 0 < \alpha < \frac{\pi}{2}. \tag{1}$$

Without loss of generality we can choose $|u\rangle = |+\rangle$, i.e. to be a spin ½ state in the $+z$ direction, and $|v\rangle$ and $|w_n\rangle$ to be spin ½ states in the $+xz$ half-plane at angles of 2α and $\pi - 2\alpha$ with the $+z$ direction, respectively. Note that $|w_-\rangle$ is the state orthogonal to $|v\rangle$, i.e. $\langle w_-|v\rangle = 0$. In each repetition of an *n* experiment, Alice produces either the spin state $|u\rangle_A$ or $|v\rangle_A$ and Bob produces either the spin state $|u\rangle_B$ or $|w_n\rangle_B$. The spins are allowed to interact via the Hamiltonian $H_n = \boldsymbol{\sigma}^A.\mathbf{D}_n.\boldsymbol{\sigma}^B$ where the components of $\boldsymbol{\sigma}^A$ and $\boldsymbol{\sigma}^B$ are the Pauli matrices and $\mathbf{D}_n$ is a second rank tensor. As mentioned, the values n = ±1 require different experimental set-ups. We choose the Hamiltonian

$$H_n = a\sigma_x^A\sigma_x^B + b\sigma_y^A\sigma_y^B + c\sigma_z^A\sigma_z^B + d\left(\sigma_x^A\sigma_z^B - n\sigma_z^A\sigma_x^B\right) \tag{2}$$

where *a*, *b*, *c* and *d* are real, dimensionless constants (with energy in appropriate units). The eigenvalues and eigenstates of H_n are

$$\begin{aligned}
E_1 &= -n(a-b)+c, & |e_1\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle_A|+\rangle_B - n|-\rangle_A|-\rangle_B\right)\\
E_2 &= n(a+b)-c, & |e_2\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle_A|-\rangle_B + n|-\rangle_A|+\rangle_B\right)\\
E_3 &= -n\left(b+\sqrt{(a+nc)^2+4d^2}\right), & |e_3\rangle &= \frac{1}{\sqrt{2}}\left(\cos\varphi\left(|+\rangle_A|+\rangle_B + n|-\rangle_A|-\rangle_B\right)+\sin\varphi\left(|+\rangle_A|-\rangle_B - n|-\rangle_A|+\rangle_B\right)\right)\\
E_4 &= -n\left(b-\sqrt{(a+nc)^2+4d^2}\right), & |e_4\rangle &= \frac{1}{\sqrt{2}}\left(-\sin\varphi\left(|+\rangle_A|+\rangle_B + n|-\rangle_A|-\rangle_B\right)+\cos\varphi\left(|+\rangle_A|-\rangle_B - n|-\rangle_A|+\rangle_B\right)\right)
\end{aligned} \tag{3}$$

where $\tan\varphi = \frac{1}{2d}\left(a+nc+\sqrt{(a+nc)^2+4d^2}\right)$. All the eigenstates are non-degenerate provided $a \neq nc$. If we choose the spin-spin interaction so that $\tan\varphi = n\cot\alpha$, each of the four states which can be produced by Alice and Bob are orthogonal to one of the $|e_i\rangle$:

$$\begin{aligned}
&|u\rangle_A|u\rangle_B \text{ is orthogonal to } |e_2\rangle, \qquad |u\rangle_A|w_n\rangle_B \text{ is orthogonal to } |e_3\rangle,\\
&|v\rangle_A|u\rangle_B \text{ is orthogonal to } |e_4\rangle \text{ and } |v\rangle_A|w_n\rangle_B \text{ is orthogonal to } |e_1\rangle.
\end{aligned} \tag{4}$$

It follows from the argument in the previous section that *u* is disjoint with *v* and/or w_n for both values of n = ±1 (from separate experiments) which proves the Three-State Theorem. The states $|w_+\rangle$ and $|v\rangle$ are the same state in this construction if $|\langle v|u\rangle| = 1/\sqrt{2}$ ($\alpha = \pi/4$ in Eq. (1)) so we have also proved the Two-State (PBR) Theorem for any two states $|u\rangle$ and $|v\rangle$ for which $|\langle v|u\rangle| = 1/\sqrt{2}$.

4. Experimental implementation

The key requirements to implement the present protocol are (i) manipulate the spins of Alice and Bob into the required states, (ii) interact the spins via H_n in Eq. (2) and

(iii) determine which of the eigenstates $|e_i\rangle$ is occupied. One possibility is a pair of two-level atoms in an optical cavity. The dipole-dipole interaction between such a pair was involved in a recent investigation [4].

For n = ±1, the interaction between the spins described by Eq. (2) can result from the standard dipole-dipole interaction if the spins are separated by a distance with direction cosines $(\sin\theta, 0, \cos\theta)$ [in the same co-ordinate system used for the spin states]. Then, in the chosen units, $a = 3\sin^2\theta - 1$, $b = -1$, $c = 3\cos^2\theta - 1$ and $d = 3\sin\theta\cos\theta$. The condition for the orthogonality of states in Eq. (4) requires

$$2\cot 2\theta = \cot\alpha - \tan\alpha. \tag{5}$$

For example, if the state $|v\rangle$ is in the +x-direction, the line joining the spins should be at $\theta = 45^\circ$ to the z-axis. If Eq. (5) cannot be satisfied or n = ±1, a conventional dipole-dipole interaction is not sufficient but there is no reason in principle that the required values could not be achieved because there are nine independent components of $\mathbf{D}_n$ in a sufficiently low symmetry environment [5].

5. Discussion and conclusion

PBR have shown that the statistical interpretation does not apply to any two states $|u\rangle$ and $|v\rangle$. The Three-State Theorem proposed here shows that if the statistical interpretation applies to $|u\rangle$ and $|v\rangle$, it does not apply to $|u\rangle$ and the state orthogonal to $|v\rangle$ (nor to $|u\rangle$ and another state $|w_+\rangle$). Also, the Three-State Theorem yields the PBR result when $|\langle u|v\rangle|^2 = ½$.

The PBR and present approaches are compared in Table 1. One advantage of the present result is that the proof, and experimental verification, involves just two independent sources of the states even when $|u\rangle$ and $|v\rangle$ are nearly the same state, i.e. $|\langle u|v\rangle| \approx 1$. The PBR theorem requires a large number N of independent sources of the states in each run of the experiment to verify the result when $|\langle u|v\rangle| \approx 1$. As N becomes larger, the experimental error that can be tolerated becomes smaller. It is also an advantage that the present approach involves only the interaction between pairs of spins and an energy measurement while PBR requires an N-bit entangling gate, which must operate without post-selection to avoid the possibility of re-distribution of the physical state λ.

Table 1. Comparison of the Two-State (PBR) and Three-State (Present) methods.

Method, key experimental requirement	$N = 2$			$N > 2$
	$\lvert\langle u\lvert v\rangle\rvert < 0.5$	$\lvert\langle u\lvert v\rangle\rvert = 0.5$	$\lvert\langle u\lvert v\rangle\rvert > 0.5$	$\lvert\langle u\lvert v\rangle\rvert$ any
PBR, N-bit entangling gate without using post-selection, low error toleration for large N	Two-State Theorem	Two-State Theorem	No result	Two-State Theorem
Present, spin-spin interaction	Three-State Theorem	Two-State Theorem	Three-State Theorem	N/A

References

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