*Topic area:* **CMMSP**

# Self-consistent linear response approximation for longitudinal and transverse plasmons

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## Introduction

The study of the longitudinal plasmons and the transverse plasmons has been playing a significant role in theoretical condensed matter physics since the work of Bohm and Pines [1]. One of the most well-known and successful theoretical methods to calculate the dispersion relation of the longitudinal plasmons is the random phase approximation (RPA). However, the RPA seems not suitable to calculate the dispersion relation of the transverse plasmons. As the transverse plasmon is nothing else but a propagating electromagnetic wave in a many-electron system, the calculation of the dispersion requires the dynamics of the electromagnetic field as well as the dynamics of the electrons. A straightforward and powerful way to perform such a calculation is the self-consistent linear response approximation (SCLRA) [2]. In the SCLRA, the longitudinal and the transverse plasmons can be treated within the same formulation. In this paper we present a unified derivation of the dispersion relations of the longitudinal and the transverse plasmons.

## SCLRA equation

In quantum many-body field theory, the many-electron system is described by the second quantized Schrödinger fields that satisfy the equal-time canonical anti-commutation relation

$$\psi_\alpha(\mathbf{x},t)\psi^\dagger_\beta(\mathbf{x}',t)+\psi^\dagger_\beta(\mathbf{x}',t)\psi_\alpha(\mathbf{x},t)=\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}'), \tag{1}$$

where the Greek subscripts denote spin variables. We assume an external electromagnetic field perturbs the system. In the linear response theory, expectation values of transverse electric current density and charge density are given by linear response formulae

$$\langle J^t_\mu(\mathbf{x},t)\rangle_A=\langle J^t_\mu(\mathbf{x},t)\rangle+\frac{4\pi e^2}{\hbar^2}\int_{-\infty}^{\infty}d^3\mathbf{x}'\int_{-\infty}^{\infty}dt'\Lambda^t_{\mu\nu}(\mathbf{x}-\mathbf{x}',t-t')A^t_\nu(\mathbf{x}',t'), \tag{2}$$

$$\langle q(\mathbf{x},t)\rangle_A=\langle q(\mathbf{x},t)\rangle+\frac{e^2}{\hbar}\int_{-\infty}^{\infty}d^3\mathbf{x}'\int_{-\infty}^{\infty}dt'D(\mathbf{x}-\mathbf{x}',t-t')\Phi(\mathbf{x}',t'), \tag{3}$$

where $\Lambda^t_{\mu\nu}$ is the transverse current-current response function defined as

$$\Lambda^t_{\mu\nu}(\mathbf{x}-\mathbf{x}',t-t')\equiv-i\theta(t-t')\langle[j^t_\mu(\mathbf{x},t),j^t_\nu(\mathbf{x}',t')]\rangle, \tag{4}$$

and $D$ is the density-density response function defined as

$$D(\mathbf{x}-\mathbf{x}',t-t')\equiv-i\theta(t-t')\langle[\rho(\mathbf{x},t),\rho(\mathbf{x}',t')]\rangle. \tag{5}$$

The expectation value with the electromagnetic perturbation is denoted by $\langle \cdots \rangle_A$ and that without electromagnetic perturbation is denoted by $\langle \cdots \rangle$ defined as

$$\langle \cdots \rangle \equiv \langle \Phi_F | \cdots | \Phi_F \rangle \tag{6}$$

with the Fermi ground state $|\Phi_F\rangle$.

On the other hand, the dynamics of the electromagnetic field is determined by Maxwell's equation. The essential assumption of the SCLRA is to replace the current density and the charge density in Maxwell's equation with these given by (2) and (3). Using the SCLRA, we obtain the following equations:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] A^t_\mu(\mathbf{x},t) = -\frac{4\pi}{c}\langle J^t_\mu(\mathbf{x},t)\rangle_A, \tag{7}$$

$$-\nabla^2\Phi(\mathbf{x},t) = 4\pi\delta\langle q(\mathbf{x},t)\rangle, \tag{8}$$

with induced charge density

$$\delta\langle q(\mathbf{x},t)\rangle \equiv \langle q(\mathbf{x},t)\rangle_A - \langle q(\mathbf{x},t)\rangle. \tag{9}$$

We have adopted the Coulomb gauge. These equations, (7) and (8), are called the SCLRA equations.

## Derivation of the transverse plasmon mode

In this section we shall derive the dispersion relation of the transverse plasmon from the SCLRA equation for the vector potential (7). Combining the Fourier transform of (7) with the Fourier transform of (2), we get

$$\omega^2 = \omega_{pl}^2 + c^2k^2 + \frac{4\pi e^2}{\hbar}\{\Re\Lambda^t(\mathbf{k},\omega) + i\Im\Lambda^t(\mathbf{k},\omega)\}, \tag{10}$$

where $\omega_{pl}$ is plasma frequency. The function $\Lambda^t$ is given by

$$\Lambda^t(\mathbf{k},\omega) \;\equiv\; \frac{\hbar^2}{4\pi^2 m^2}\int_0^\infty dp\int_{-1}^{1} d(\cos\theta)\theta(k_F-p)p^4(1-\cos^2\theta)g(pk\cos\theta) \tag{11}$$

with the function $g$ defined as

$$g(pk\cos\theta) \;\equiv\; \frac{1}{\omega-(\omega_{\mathbf{p}+\mathbf{k}}-\omega_{\mathbf{p}})+i\eta} - \frac{1}{\omega+(\omega_{\mathbf{p}+\mathbf{k}}-\omega_{\mathbf{p}})+i\eta}, \tag{12}$$

and the step function $\theta(k_F-p)$. Calculating the integral in (11), we obtain respectively the real and the imaginary part of $\Lambda^t$:

$$\begin{aligned}\Re\Lambda^t(\mathbf{k},\omega) \;=\; \frac{\hbar k_F^4}{4\pi^2 mk}\Bigg[&\frac{5}{6}v_- - \frac{1}{2}v_-^3 + \frac{1}{4}(1-v_-^2)^2\log\left|\frac{1+v_-}{1-v_-}\right| \\ &-\frac{5}{6}v_+ + \frac{1}{2}v_+^3 - \frac{1}{4}(1-v_+^2)^2\log\left|\frac{1+v_+}{1-v_+}\right|\Bigg],\end{aligned} \tag{13}$$

$$\Im\Lambda^t(\mathbf{k},\omega) \;=\; \frac{\hbar k_F^4}{16\pi mk}\left[(1-v_+^2)^2\theta(1-|v_+|)-(1-v_-^2)^2\theta(1-|v_-|)\right], \tag{14}$$

where we have defined

$$v_\pm \equiv \frac{m\omega}{\hbar k_F k} \pm \frac{k}{2k_F}. \tag{15}$$

The imaginary part of (10) gives the damping of propagating transverse electromagnetic fields in the electron gas. In order to derive the transverse plasmon dispersion, we consider small $k$ in the domain $|v_\pm| > 1$. Then, the imaginary part (14) vanishes, and the real part (13) becomes

$$\Re\Lambda(\mathbf{k},\omega) = \frac{\hbar^3 k_F^5 k^2}{15\pi^2 m^3\omega^2} + \frac{\hbar^5 k_F^7 k^4}{35\pi^2 m^5\omega^4} + O(k^6). \tag{16}$$

Substituting (16) into (10), we obtain the dispersion relation of the transverse plasmon

$$\omega^2 \;=\; \omega_{pl}^2 + c^2k^2 + \omega_{pl}^2\left[\frac{1}{5}\left(\frac{\hbar k_F k}{m\omega}\right)^2 + \frac{3}{35}\left(\frac{\hbar k_F k}{m\omega}\right)^4\right] + O(k^6). \tag{17}$$

This is the condition for the propagating transverse electromagnetic field in the electron gas. Using this dispersion relation, we can calculate the index of refraction and the penetration depth [3].

## Derivation of the longitudinal plasmon mode

In this section we shall derive the dispersion relation of the longitudinal plasmon. Combining the Fourier transform of (8) with the Fourier transform of (3), we get

$$\frac{4\pi e^2}{\hbar k^2}\{\Re D(\mathbf{k},\omega) + i\Im D(\mathbf{k},\omega)\} = 1. \tag{18}$$

The imaginary part in (18) gives the damping of longitudinal electric fileds. Calculating the Fourier transform of (6), we obtain

$$\begin{aligned}\Re D(\mathbf{k},\omega) \;=\; \frac{mk_F^2}{2\pi^2\hbar k}\Bigg[& v_- + \frac{1}{2}(1-v_-^2)\log\left|\frac{1+v_-}{1-v_-}\right| \\ & -v_+ - \frac{1}{2}(1-v_+^2)\log\left|\frac{1+v_+}{1-v_+}\right|\Bigg],\end{aligned} \tag{19}$$

$$\Im D(\mathbf{k},\omega) = \frac{mk_F^2}{4\pi\hbar k}\left[(1-v_+^2)\theta(1-|v_+|)-(1-v_-^2)\theta(1-|v_-|)\right], \tag{20}$$

We consider small $k$ in the domain $|v_\pm| > 1$. Then, the imaginary part (20) vanishes, and the real part (19) becomes

$$\Re D(\mathbf{k},\omega) = \frac{mk_F}{3\pi^2\hbar}\left[\left(\frac{\hbar k_F k}{m\omega}\right)^2 + \frac{3}{5}\left(\frac{\hbar k_F k}{m\omega}\right)^4 + O(k^6)\right]. \tag{21}$$

Substituting (21) into (18), we obtain the longitudinal plasmon dispersion

$$\omega^2 = \omega_{pl}^2\left[1 + \frac{3}{5}\left(\frac{\hbar k_F k}{m\omega}\right)^2\right] + O(k^4). \tag{22}$$

This result agrees with the RPA result.

**Conclusion**

We have derived the dispersion relations of the longitudinal and the transverse plasmon. For the longitudinal plasmon mode, we showed that RPA result can be obtained within the framework of the SCLRA. As we have shown, the advantages of the SCLRA are its universality to allow a unified treatment of both longitudinal and transverse plasmons on the same theoretical footing, and also its clear physical meaning. Furthermore, the SCLRA is most suited to be incorporated with the non-perturbative canonical formulation of quantum many-body theory [4] [5].

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