Critical exponents for structural transitions in a complex plasma

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Abstract

The critical instability of two dust particles levitating in the complex plasma sheath of a radio-frequency discharge is considered. It is shown that the two-particle system has a critical point where the alignment symmetry is continuously broken as the system parameter is varied. The associated critical exponents are derived and found to belong to the Ising universality class. Another universality class is suggested for symmetry breaking of the confinement in the horizontal and vertical directions.

Introduction

The amazing property of critical phenomena is their universality when similar scaling appears in different systems: e.g., magnets and gases follow simple power laws for the order parameter, specific heat capacity, susceptibility, compressibility, etc [1]. In thermodynamic systems, phase transitions take place at a critical temperature when the coefficients that characterize the linear response of the system to external perturbations diverge [2]. The corresponding theory of critical phenomena has mostly been explored from the perspective of the statistical thermodynamics [3]. In so-called extensive systems, the number of interacting particles is of the order of Avogadro's number, so the assumption of an infinite uniform system is justified. In non-extensive systems where the number of particles is much lower, the thermodynamic limit cannot be applied, since the extent of the particle interaction is comparable with the size of the system. The recent prediction of a liquid-vapour critical point in an extensive type complex plasma [4] has sparked interest in the possible universality near the critical point. The question whether the universal scaling also takes place in these systems is still open. Complex plasmas [5, 6] provide an ideal medium for studying phase transitions in non-extensive systems, when even the system of two particles displays rich physics [7, 8, 9, 10]. Here, the critical instability of two dust particles levitating in the complex plasma sheath of a radio-frequency discharge is considered. It is shown that the two-particle system has a critical point where the alignment symmetry is continuously broken as the system parameter is varied. The associated critical exponents are derived and found to belong to the Ising universality class. Another universality class is suggested for symmetry breaking of the confinement in the horizontal and vertical directions.

Results

A non-extensive system can be represented in general canonical form by the finite-dimensional, second-order differential equation

$$M\ddot{\boldsymbol{q}} + B\dot{\boldsymbol{q}} + C\boldsymbol{q} = \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

where the matrices B and C are in general non-symmetric and $\boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is the non-linear term. Let $\boldsymbol{q}_0$ be an equilibrium subject to the condition $\dot{\boldsymbol{q}} = \boldsymbol{0}$. It suffices to take $\boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{N}$ since by Lyapunov's theorem, spectral instability implies non-linear instability. The equation of linearized oscillations about $\boldsymbol{q}_0$ is thus

$$M\ddot{\boldsymbol{q}} + (C_0 + P)\boldsymbol{q} = \boldsymbol{N}.$$

where C_0 and P are the symmetric and skew-symmetric matrices corresponding the the potential and non-potential (positional) components of force, respectively. For a system of two dust particles,

$$M\ddot{\boldsymbol{q}} + (M\Omega^2 + D + W)\boldsymbol{q} = \boldsymbol{N}.$$

where $\boldsymbol{q}$ are the generalized coordinates of the dust particles, Ω is the diagonal matrix of resonant angular frequences of a lone particle in the confinement well and $-D\boldsymbol{q}$ and $-W\boldsymbol{q}$ are the linearized forces due to the interparticle Debye interaction, and the non-reciprocal wake-particle interactions, respectively.

In one of the simplest approximations, the wake is represented by an excessive positive point charge Q_w, located at a certain distance ℓ below the dust particle. In this case, the wake-potential for each particles has the form $\Phi_\mathrm{w} = (Q_\mathrm{w}/4\pi\epsilon_0)\exp(-\kappa\Delta_\mathrm{w})/\Delta_\mathrm{w}$ where $\Delta_\mathrm{w} \equiv \sqrt{(\Delta x)^2 + (\Delta z + \ell)^2}$. Note that the asymmetric charge polarisation surrounding the dust particle breaks the reflectional $\Delta z \to -\Delta z$ symmetry of the wake-field, resulting in a non-symmetric coefficient matrix C. In the limit as $\Delta z \to 0$, however, the antisymmetric terms vanish leaving a purely potential force. This suggests that asymptotically close to the horizontal plane, the system may be regarded as Hamiltonian.

The Order Parameter Exponent β.
Below a critical radial confinement $\omega_{\rho,\mathrm{c}}$, the Hamiltonian has a stable equilibrium with the dust particles horizontally aligned ($\Delta z = 0$). Following Lampe *et al.* [11], we write potential part of the Hamiltonian in terms of the interparticle separations $(\Delta x, \Delta z)$. Despite not being true for the full range of angles, the Hamiltonian approximation is valid near the horizontal plane where the critical point occurs. From this approximately isotropic position in the field, the potential energy of the two-particle system can only depend upon the coordinate differences. Taylor expanding the effective potential about the equilibrium position in the vertical interparticle separation order parameter Δz we obtain

$$\Pi = \Pi'(0)\Delta z + \frac{1}{2}\Pi''(0)(\Delta z)^2 + \frac{1}{6}\Pi^{(3)}(0)(\Delta z)^3 + \frac{1}{24}\Pi^{(4)}(0)(\Delta z)^4 = -\frac{1}{2}M\omega_z^2\zeta(\Delta z)^2 + \frac{1}{4}a_4(\Delta z)^4 \qquad (1)$$

where we have written $a_4 \equiv \frac{1}{6}\Pi^{(4)}(0)$. Equation (1) corresponds to the Ginzburg-Landau Hamiltonian for the Ising model in the mean field approximation with zero external magnetic field strength $H = 0$ [2]. The equilibria of (1) are found using the minimum condition $\mathrm{d}\Pi/\mathrm{d}\Delta z = 0$,

$$0 = -M\omega_z^2\zeta\Delta z + a_4(\Delta z)^3. \qquad (2)$$

For $\zeta < 0$, the only real solution is $\Delta z = 0$. That is, below the critical frequency, the dust particles remain aligned in the horizontal plane. Above the critical frequency $\zeta \geq 0$, the ground-state of the system bifurcates into two degenerate equilibria which are related by the $\Delta z \rightarrow -\Delta z$ symmetry of the Hamiltonian

$$\Delta z = \pm\sqrt{\frac{M\omega_z^2\zeta}{a_4}}. \qquad (3)$$

The order parameter changes continuously as the frequency passes the critical point, with the critical exponent $\beta = 1/2$ defined by $\Delta z \propto |\zeta|^\beta$ characteristic of a continuous, or second-order transition. The Ising model of ferromagnetics undergoes such a transition from paramagnetic to ferromagnetic phase as $T \rightarrow T_\mathrm{c}^+$ at zero external magnetic field strength. In the case of the Ising model, the zero-field magnetization $M|_{H=0} \propto |\epsilon|^{1/2}$ serves as the order parameter. At non-zero magnetic field strength, the system loses its spin-reversal symmetry, so that the continuous phase transition can no longer occur. The external magnetic field H is said to be conjugated to the order parameter M.

The Response Exponent δ.
Motivated by the external magnetic field of the Ising model, we search for a field conjugated to the order parameter Δz of the dust system. The asymmetric wake provides this field. Introducing an asymmetry between the wake charges of two particles ΔQ_w induces explicit symmetry breaking terms in the Hamiltonian

$$\Pi = a_1\Delta Q_\mathrm{w}\Delta z - \frac{1}{2}M\omega_z^2\zeta(\Delta z)^2 + \frac{1}{3}a_3\Delta Q_\mathrm{w}(\Delta z)^3 + \frac{1}{4}a_4(\Delta z)^4. \qquad (4)$$

Note that it is necessary to replace each instance of Q_w in the expressions for the coefficients a_1 and a_4 by the sum of both wake charges, which is assumed to remain constant. The symmetry breaking terms skew the Hamiltonian so that the oblique equilibria lose their degeneracy and one becomes energetically favored. Along the critical isofrequency $\zeta = 0$, the equilibrium condition gives

$$\Delta Q_\mathrm{w} = -\frac{a_4(\Delta z)^3}{a_1 + a_3(\Delta z)^2}. \qquad (5)$$

The first term in the Taylor expansion of (5) about $\Delta z = 0$ occurs at third order in Δz, and all subsequent terms occur at fifth order or higher. This provides the critical exponent $\delta = 3$ for the scaling relation $\Delta z \propto \Delta Q_\mathrm{w}^{1/\delta}$. If the wake field ΔQ_w passes zero at or above the critical frequency, then the dust system will change continuously from one oblique equilibrium to the other. If this occurs below the critical frequency, the order parameter Δz develops a singularity at $\Delta Q_\mathrm{w} = 0$, resulting in a discontinuous jump in the order parameter, known as a first order transition.

The Susceptibility Exponent γ.
The susceptibility χ of a system is defined as the linear response of the order parameter to infinitesimal changes in

the conjugate field. The critical exponent γ is defined such that $\chi = (\partial M/\partial H)_{H\to 0} \propto |\epsilon|^{-\gamma}$. In the mean field approximation, the predicted value is $\gamma = 1$

The equilibria of the general Hamiltonian of the dust system (4) are analytically soluble. Differentiating the solution with respect to the wake difference $\Delta Q_{\rm w}$ and taking the limit of the derivative as $\Delta Q_{\rm w} \to 0$ we obtain the critical exponent $\gamma = 1$ for the susceptibility $\chi = (\partial \Delta z/\partial \Delta Q_{\rm w})_{\Delta Q_{\rm w}\to 0} \propto |\zeta|^{-\gamma}$.

The Specific Heat Exponent α.
This exponent describes the divergence of the specific heat $C \propto |\epsilon|^{-\alpha}$ at the critical temperature. The value $\alpha = 0$ follows trivially from the definition of the specific heat of the dust system, $C = (\partial \Pi/\partial \zeta) \propto |\zeta|^{-0}$.

Thus the critical exponents for the horizontal alignment instability are the same as those of the mean field theory for thermodynamic systems such as the Ising model and Van der Waals theory. The dust system exhibits both a continuous (second order) transition at the critical resonant frequency, as well as discontinuous transitions induced by asymmetric wake fields.
The exponents are independent of the plasma parameters such as the Debye length. Although we made use of the point-charge approximation to model the ion wake distribution, the results should apply equally well for any wake distribution since they depend only on the local approximation to wake potential up to third order. The non-extensive dust system is unlike other mean field models in that the system is only asymptotically Hamiltonian in the order parameter Δz. This does not affect the resulting critical exponents since the location of the first stage of the transition coincides with the limit in which the system may be regarded as Hamiltonian.

Consider now symmetry breaking of a different kind, namely the breaking of confinement symmetry in the radial and vertical directions. The vertical ion flow in the sheath naturally provides a prevalent direction, reducing the symmetry of the system. In terms of the model, radial/vertical confinment symmetry is affected by two independent parameters: the ratio of the confinment angular frequencies and the strength of the wake-field $Q_{\rm w}$, with perfect radial/vertical confinement symmetry characterised by no energy barrier to rotation in the vertical plane. To describe the extent of the symmetry breaking by the wake-field, the critical ratio of radial to vertical sheath frequencies is used as an order parameter. The equation of linearized oscillations about the horizontal plane gives the following general expression for the critical ratio of radial to vertical confinement frequencies [8]

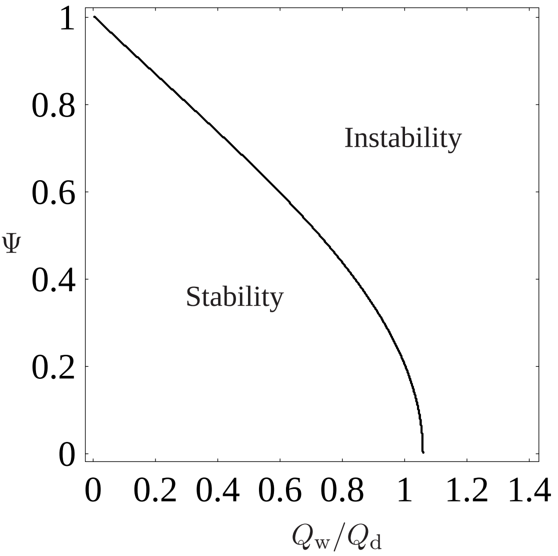

Figure 1: The numerical phase diagram ($\lambda_{\rm D} = 6.07\times 10^{-4}\,$m, $M = 3.60\times 10^{-14}\,$kg, $Q_{\rm d} = 3.6\times 10^{3}e$, $\ell = 1.5\lambda_{\rm D}$)

$$\Psi^2 = 1 - \frac{Q_{\rm w}Q_{\rm d}}{2\pi\epsilon_0 M\omega_z^2}\left[\frac{\ell^2\kappa}{\Delta_{\rm w}^4} - (1+\kappa\Delta_{\rm w})\left(\frac{3\ell^2}{\Delta_{\rm w}^5} + \frac{\ell^2\kappa}{\Delta_{\rm w}^4}\right)\right] {\rm e}^{-\kappa\Delta_{\rm w}}. \tag{6}$$

Plotting Ψ as a function of $Q_{\rm w}$ at constant ω_z as in Fig. 1, reveals a curve which terminates at a critical wake charge $Q_{\rm w,c}$. The unexpectedly high values of the wake charge near the critical point could be attributable to the perturbation of the centre of mass of the dust system by the presence of the wake, which for simplicity was not considered in the stability analysis. Given that the coordinates of the critical point in the $(Q_{\rm w}, \Psi)$ space are $(Q_{\rm w,c}, 0)$, it quickly follows that near the critical point

$$\Psi = \pm\sqrt{\frac{Q_{\rm w,c} - Q_{\rm w}}{Q_{\rm w,c}}} \tag{7}$$

and thus we obtain the critical exponent $\beta = 1/2$ defined by $\Psi \sim (-\theta)^{\beta}$, where $\theta \equiv (Q_{\rm w} - Q_{\rm w,c})/Q_{\rm w,c}$.

If we define the "spatial susceptibility" by

$$\chi_{\rm spat} \equiv \left(\frac{\partial\omega_\rho}{\partial\omega_z}\right)_{\omega_z\to 0} \tag{8}$$

then, multiplying (6) by ω_z, partially differentiating with respect to ω_z and taking the limit as $\omega_z \to 0$ we obtain the critical exponent $\gamma = 1/2$ where $\chi_{\rm spat} \sim (-\theta)^{-\gamma}$.

To assist in the determination of the response exponent δ, we introduce the normalized resonant frequencies $\widetilde{\omega}_\rho \equiv \omega_\rho/\omega_{z,{\rm c}}$ and $\widetilde{\omega}_z \equiv \omega_z/\omega_{z,{\rm c}}$. In this notation, (6) becomes

$$\Psi = \sqrt{1 - \frac{\theta+1}{\widetilde{\omega}_z^2}}. \tag{9}$$

As the critical point is approached, we may write

$$\widetilde{\omega}_z = \frac{1}{\sqrt{1-\widetilde{\omega}_\rho^2}} \approx 1 + \frac{1}{2}\widetilde{\omega}_\rho^2$$

and therefore $\delta = 2$. The critical exponents $\beta = 1/2$, $\gamma = 1/2$ and $\delta = 2$ satisfy the Widom equality$\gamma = \beta(\delta - 1)$

suggestive of universality. Assuming that the Griffith's equality $\alpha + \beta(\delta + 1) = 2$ holds true, we may assign the critical exponent $\alpha = 1/2$ to describe the divergence of the specific heat capacity $C = \partial Q/\partial\theta \propto |\theta|^{-\alpha}$.

The second order transition can be easily observed for two identical particles in a uniform discharge. In order to observe the discontinuous first order transition between the oblique equilibria, consideration must be given to the experimental realization of the wake charge asymmetry $\Delta Q_{\rm w}$. The wake asymmetry may come about by virtue of the non-uniformity of the discharge, however, the dynamical wake charging in the plasma is necessary to experimentally observe the jump.

In terms of spatial symmetry breaking of the confinement fields, the critical frequency ratio Ψ for two identical dust particles provides an order parameter to describe the extent of the symmetry breaking by the dust-induced wake fields. The remarkable simplicity of the order parameter scaling $\Psi \sim (-\theta)^{1/2}$ near the critical wake charge tempts us to define other critical exponents to describe the susceptibility and the response. The agreement exponents thus defined with the Widom equality gives strong support to the existence of an underpinning universality class. It is not clear, at present, if the predicted exponent of $\alpha = 1/2$ can be derived from the Hamiltonian.

To conclude, we have suggested that the critical point of the horizontal alignment instability belongs to the Ising universality class for the mean field thermodynamic phase transitions. A new universality class has been proposed for the breaking of the external confinement symmetry. The role of fluctuations has not been considered. Thermal fluctuations in thermodynamic systems are responsible for changing the critical exponents from their mean field values. Fluctuations of the grain charge, for example, in the non-extensive dust system may allow us to define a correlation function, giving deeper insight into the universality class of this system. Finally, the non-potentiality of the interaction forces needs further attention. In particular, any additional correction they may induce in the critical exponents far from the critical point needs to be explored.

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