CHARACTERISTICS OF THE TRAJECTORY OF A PROJECTILE IN A LINEAR RESISTING MEDIUM AND THE LAMBERT W FUNCTION

Seán M. Stewart
Core Program, The Petroleum Institute, Abu Dhabi, United Arab Emirates

Abstract
The well-known problem of a projectile in a linear resisting medium is re-examined in light of the recently defined Lambert W function. Using closed-form analytic expressions for the time of flight, range, and optimal angle that maximises the range expressed in terms of the Lambert W function, a number of features characteristic to the trajectory of a projectile in a linear resisting medium are confirmed, or established for the first time, using elementary means.

Introduction
The problem of a projectile moving in a resisting medium such as air is a very old one having first received serious attention in the mid to late seventeenth century (Groetsch and Cipra 1997). Historically the problem was of great importance since it was intimately connected with one of the most pressing problems of its day, the study of ballistics. Fortunately nowadays more peaceful interest in projectiles moving through resisting media is to be found in applications such as those associated with various sports (Linthorne 2001; Linthorne *et al.* 2005; Linthorne and Everett 2006) or in the modelling of particle processes in water or air (Brown and Lawler 2003), among others.

Until quite recently, inclusion of a resistive force on the motion of a projectile as it moves through a resisting medium resulted in equations of motion that could not be completely solved in closed analytic form for those quantities of physical interest such as the time of flight, range, and optimal angle of projection of the projectile. Instead one was forced to resort to a number of different approximation schemes valid within various limiting cases of interest (de Mestre 1990). The emergence in recent years of a new 'implicitly' elementary function now known as the Lambert W function[1] (Corless *et al.* 1996) has however allowed the well-known case of a projectile in a linear resisting medium to be re-examined. In this paper the new found analyticity in expressions for the time of flight, range, and optimal angle of projection in terms of the Lambert W function will be used to establish a number of features characteristic to the projectile's trajectory. In particular, the classically known result of the difference between the descent and ascent times increasing with increasing initial launch speeds is confirmed analytically in a direct manner while the progressively more forwardly skewed behaviour in the trajectory with increasing launch speeds, while often intuitively thought to be true, is rigorously established. It is also established that for a fixed initial launch speed in a given medium the forward blunting behaviour in the trajectory of the projectile becomes maximised at an angle greater than the corresponding optimal angle which maximises its range.

Linear resisted projectile motion
Consider an object of mass m that is launched with initial speed v_0 from the origin at an angle of α to the horizontal over level ground and in a uniform gravitational field g. In a linear resisting medium where the resistive force acting on the projectile is taken to be proportional to its velocity, on solving the resulting equations of motion in the usual way, one obtains

$$x(t) = \frac{v_0 \cos\alpha}{\gamma}(1 - \mathrm{e}^{-\gamma t}) \quad \text{and} \quad y(t) = \left(\frac{g}{\gamma^2} + \frac{v_0 \sin\alpha}{\gamma}\right)(1 - \mathrm{e}^{-\gamma t}) - \frac{gt}{\gamma}. \tag{1}$$

Here γ is a fixed positive drag coefficient per unit mass while $x(t)$ and $y(t)$ correspond to the horizontal and vertical displacements of the projectile at time t, respectively.

The time of flight T for the projectile is found by setting $y(T) = 0$. On doing so, after having rearranged terms algebraically, one obtains

$$u = \zeta(\alpha)(1 - \mathrm{e}^{-u}). \tag{2}$$

Here $u = \gamma T$, $\zeta(\alpha) = 1 + c \sin\alpha$ while $c = \gamma v_0 / g$. Although (2) is traditionally thought of as a transcendental equation, recently it has become possible to solve such an equation in closed-form in terms of the Lambert W function (Stewart 2005a; Stewart 2005b).

[1] It is included as an in-built library function in many computer algebra systems. For example, `LambertW[x]` in Maple and `ProductLog[x]` in Mathematica.

Briefly, the Lambert W function, $\mathrm{W}(x)$, is defined to be the function which solves the transcendental equation

$$\mathrm{W}(z)\mathrm{e}^{\mathrm{W}(z)} = z, \quad z \in \mathbb{C}. \tag{3}$$

Referred to as the *defining equation* for the Lambert W function, (3) has infinitely many solutions (most of which are complex) and is therefore multivalued. It is usual to write $\mathrm{W}_k(z)$ where $k = 0, \pm 1, \pm 2, \ldots$ denotes the branch index for the function. When z is real, only two branches of $\mathrm{W}_k(z)$ take on real values. They are the principal branch $\mathrm{W}_0(z)$ and the secondary real branch $\mathrm{W}_{-1}(z)$ and are real for $z \in [-\mathrm{e}^{-1}, \infty)$ and $z \in [-\mathrm{e}^{-1}, 0)$, respectively. The branch point between the two real branches occurs at $z = -\mathrm{e}^{-1}$.

Returning to (2), the transcendental equation can now be readily solved for u, and hence T, in terms of the Lambert W function. The result is (Stewart 2005a)

$$T(\alpha) = \frac{v_0 \sin\alpha}{g}\left(\frac{\zeta(\alpha) + \mathrm{W}_0(-\zeta(\alpha)\mathrm{e}^{-\zeta(\alpha)})}{\zeta(\alpha) - 1}\right). \tag{4}$$

The principal branch is chosen as it is the branch of the Lambert W function which gives the non-trivial solution to (2). A closed-form expression for the range R of a projectile in a linear resisting medium follows immediately. On substituting for the time of flight into the horizontal component of (1) yields (Warburton and Wang 2004; Packel and Yuen 2004; Stewart 2005a)

$$R(\alpha) = \frac{v_0^2 \sin 2\alpha}{2g}\left(\frac{\zeta(\alpha) + \mathrm{W}_0(-\zeta(\alpha)\mathrm{e}^{-\zeta(\alpha)})}{\zeta(\alpha)(\zeta(\alpha) - 1)}\right). \tag{5}$$

Finally, having expressed the range of a projectile in a linear resisting medium in closed-form in terms of a now known analytic function allows the angle $\alpha_{\max}$ which maximises the range of the projectile to be expressed as a function of the dimensionless drag coefficient c in terms of the Lambert W function. The result is (Packel and Yuen 2004; Stewart 2005a)

$$\alpha_{\max} = \sin^{-1}\left[\frac{c}{\exp\left(\mathrm{W}_0\left(\frac{c^2-1}{\mathrm{e}}\right) + 1\right) - 1}\right], \quad c > 0. \tag{6}$$

Characteristics of the trajectory on ascent and descent

The availability of analytic expressions for the time of flight and range, while having a certain intrinsic aesthetic appeal is also of great practical benefit as they allow a number of features characteristic to the trajectory of a projectile in a linear resisting medium to be established in a direct, analytic manner using all the normal tools of analysis.

For a projectile in a linear resisting medium it is well known that the time of descent t_d is greater than the time of ascent t_a (Groetsch 2003; Stewart 2006). Physically this arises from a greatly reduced vertical component in the velocity of the projectile on descent. Moreover, not only does the descent time exceed the ascent time, but as is well known this difference between the two increases with increasing initial launch speeds.

In establishing the above characteristic we recognise the sum of the ascent and descent times corresponds to the time of flight of the projectile, namely $t_d + t_a = T$. And since $t_a = v_0 \sin\alpha \ln\zeta(\alpha)/[g(\zeta(\alpha) - 1)]$, for the difference between the two one has

$$\Delta t = t_d - t_a = T - 2t_a = \frac{v_0 \sin\alpha}{g}\left(\frac{\zeta + \mathrm{W}_0(-\zeta\mathrm{e}^{-\zeta}) - 2\ln\zeta}{\zeta - 1}\right). \tag{7}$$

On differentiating with respect to the initial launch speed v_0 and simplifying, one finds

$$\frac{\mathrm{d}(\Delta t)}{\mathrm{d}v_0} = \frac{1}{g}\left(\frac{\zeta - 2 - \mathrm{W}_0(-\zeta\mathrm{e}^{-\zeta})}{\zeta[1 + \mathrm{W}_0(-\zeta\mathrm{e}^{-\zeta})]}\right). \tag{8}$$

Inequalities involving the Lambert W function have been established elsewhere (Stewart 2006). In particular, it can be shown that

$$-1 < \mathrm{W}_0(-x\mathrm{e}^{-x}) < 0 \quad \text{and} \quad x - 2 > \mathrm{W}_0(-x\mathrm{e}^{-x}) \quad \text{for} \quad x > 1.$$

Since $\zeta = 1 + \gamma v_0/g \sin\alpha > 1$ for all $\alpha \in (0, \pi/2)$ and for all $\gamma > 0$, $\mathrm{d}(\Delta t)/\mathrm{d}v_0 > 0$ for all non-zero v_0. Thus the difference in time between descent and ascent increases with increasing initial launch speeds. For projectiles in air such a result has been attributed to Marin Mersenne (1588–1648) who first observed it in 1644 (Hall 1969).

The forward skew in the trajectory of a projectile in a linear resisting medium is well known (see, e.g., Erlichson 1983). Such an asymmetry in the trajectory path is the result of the horizontal distance travelled by the projectile in its ascent to the trajectory peak x_a is greater than the corresponding horizontal distance travelled during its descent x_d and has been established analytically (Groetsch 2003; Stewart 2006). Physically this is readily understood, being the result of a decreasing horizontal component in the velocity of the projectile as it moves through the resisting medium. What has not been previously recognised before is that this difference between the two increases with increasing v_0. Thus the trajectories of projectiles in a linear resisting medium become progressively more forwardly skewed with increasing initial launch speeds.

In establishing the above characteristic, as the range of a projectile corresponds to the sum of the ascent and descent horizontal distances travelled, namely $x_a + x_d = R$, and since $x_a = v_0^2 \sin 2\alpha/[2g\zeta(\alpha)]$, the difference between the two can be written as

$$\Delta x = x_a - x_d = 2x_a - R = \frac{v_0 \cos\alpha}{\gamma}\left(\frac{\zeta - 2 - \mathrm{W}_0(-\zeta \mathrm{e}^{-\zeta})}{\zeta}\right). \tag{9}$$

On differentiating with respect to v_0, one finds

$$\frac{\mathrm{d}(\Delta x)}{\mathrm{d}v_0} = \frac{\cos\alpha}{\gamma}\left\{\frac{\zeta - 2 - \mathrm{W}_0(-\zeta\mathrm{e}^{-\zeta})}{\zeta} + \frac{(\zeta-1)(\zeta\mathrm{W}_0(-\zeta\mathrm{e}^{-\zeta})+1)}{\zeta^2(1+\mathrm{W}_0(-\zeta\mathrm{e}^{-\zeta}))} + \frac{(\zeta-1)(1+\mathrm{W}_0(-\zeta\mathrm{e}^{-\zeta}))}{\zeta^2}\right\} > 0, \tag{10}$$

for all non-zero v_0, since $\zeta\mathrm{W}_0(-\zeta\mathrm{e}^{-\zeta}) + 1 > 0$ for all $\zeta > 1$. Thus the difference in horizontal distance travelled between ascent and descent also increases with increasing initial launch speeds.

Optimal angle of projection for greatest forward skew in the trajectory

One final question we ask is, for a projectile launched with a fixed initial speed in a given linear resisting medium at what angle to the horizontal must it be launched so as to maximise the amount of forward skew observed in its trajectory. Since the horizontal distance travelled by the projectile on ascent is given by

$$x_a(\alpha) = \frac{v_0^2 \sin 2\alpha}{2g}\frac{1}{\zeta(\alpha)} = \frac{v_0^2 \sin 2\alpha}{2g(1 + c\sin\alpha)}, \tag{11}$$

the optimal angle $\alpha^*_{\max}$ of projection for greatest forward skew is found from the angle which satisfies $\mathrm{d}x_a/\mathrm{d}\alpha = 0$. Differentiating (11) with respect to α and simplifying gives

$$\frac{\mathrm{d}x_a}{\mathrm{d}\alpha} = \frac{v_0^2(1 - 2\sin^2\alpha - c\sin^3\alpha)}{g(1 + c\sin\alpha)^2}, \tag{12}$$

so that the maximum occurs when the launch angle α satisfies the cubic equation

$$cu^3 + 2u^2 - 1 = 0 \tag{13}$$

corresponding to when the numerator in (12) vanishes. Here we have set $u = \sin\alpha^*_{\max}$. Since $c > 0$ in a resisting medium *Descartes' Sign Rule* ensures the maximum number of positive real roots for the cubic is at most one. Thus there will be only one angle $\alpha^*_{\max} \in (0, \pi/2)$ for which x_a is maximised. An expression for the optimal angle is found on solving (13) using Cardano's formula. We find for the positive real root

$$\alpha^*_{\max} = \sin^{-1}\left[\frac{1}{3c}\left(\sqrt[3]{\frac{D_+}{2}} + \sqrt[3]{\frac{D_-}{2}} - 2\right)\right], \tag{14}$$

where $D_\pm = \pm 3c\sqrt{3(27c^2 - 32)} + 27c^2 - 16$.

Figure 1 shows a plot of both the optimal angle for greatest forward skew $\alpha^*_{\max}$ in the trajectory (broken line) together with the optimal angle that maximises the range $\alpha_{\max}$ (solid line) as a function of the dimensionless drag parameter $c = \gamma v_0/g$.

From the figure it is observed that the optimal angle for greatest forward skew in the trajectory is greater than the corresponding angle which maximises the range as $\alpha^*_{\max} > \alpha_{\max}$ for all $c > 0$. Finally, an interesting connection between the former angle to the golden ratio is found for the special case of $c = 1$. Here

$$\alpha^*_{\max} = \sin^{-1}\left(\frac{1}{\phi}\right), \tag{15}$$

such that $\phi = (1+\sqrt{5})/2$ corresponds to the golden ratio.

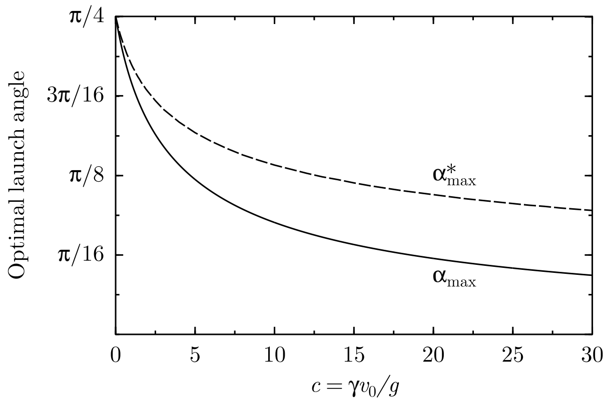

Figure 1: Optimal angle of projection for greatest forward skew (broken line) in the trajectory of a projectile in a linear resisting medium as a function of the dimensionless parameter $c = \gamma v_0/g$ compared to the corresponding angle which maximises the range of the projectile (solid line).

Conclusion

By employing the recently defined Lambert W function—a function whose importance continues to grow as it is applied with ever increasing frequency to a range of problems—a number of features characteristic to the trajectory for the well-known case of a projectile in a linear resisting medium have been established analytically using elementary means, thereby providing new insight into this classical problem.

References

Brown, P.P. and Lawler, D.F. (2003). Sphere drag and settling velocity revisited. *Journal of Environment Engineering* **129**(3), 222–231.
Corless, R.M, Gonnet, G.H, Hare, D.E.G, Jeffrey, D.J. and Knuth, D.E. (1996). On the Lambert W function. *Advances in Computational Mathematics* **5**(4), 329–359.
de Mestre, N. (1990). The mathematics of projectiles in sport. Cambridge University Press, Cambridge.
Erlichson, H. (1983). Maximum projectile range with drag and lift, with particular application to golf. *American Journal of Physics* **51**(4), 357–362.
Groetsch, C.W. and Cipra, B. (1997). Halley's comment – Projectiles with linear resistance. *Mathematics Magazine* **70**(4), 273–280.
Groetsch, C.W. (2003). Timing is everything: The French connection. *The American Mathematical Monthly* **110**(10), 950–955.
Hall, A.R. (1969). Ballistics in the seventeenth century. Harper and Row, New York, p. 107.
Linthorne, N.P. (2001). Optimum release angle in the shot put. *Journal of Sports Sciences* **19**(5), 359–372.
Linthorne, N.P., Guzman, M.S. and Bridgett, L.A. (2005). Optimum take-off angle in the long jump. *Journal of Sports Sciences* **23**(7), 703–712.
Linthorne, N.P. and Everett, D.J. (2006). Release angle for attaining maximum distance in the soccer throw-in. *Sports Biomechanics* **5**(2), 243–260.
Packel, E. and Yuen, D. (2004). Projectile motion with resistance and the Lambert W function. *The College Mathematics Journal* **35**(5), 337–350.
Stewart, S.M. (2005a). A little introductory and intermediate physics with the Lambert W function. *Proceedings of the 16th Biennial Congress of the Australian Institute of Physics*, M. Colla ed., Vol. 2, pp. 194–197. Australian Institute of Physics, Parkville, VIC.
Stewart, S.M. (2005b). Linear resisted projectile motion and the Lambert W function. *American Journal of Physics* **73**(3), 199.
Stewart, S.M. (2006). An analytic approach to projectile motion in a linear resisting medium. *International Journal of Mathematical Education in Science and Technology* **37**(4), 411–431.
Warburton, R.D.H. and Wang, J. (2004). Analysis of asymptotic projectile motion with air resistance using the Lambert W function. *American Journal of Physics* **72**(11), 1404–1407.